Homework 9

This assignment is due by **11:59pm on Friday 11/15** on gradescope. You should start a new page for each problem (if a problem has multiple parts, the parts can be completed on the same page). After completing the assignment, save your solutions as a single pdf and upload it to gradescope. Be sure to **assign your pages** on the gradescope upload page; failure to do so will result in a penalty.

You may collaborate with your peers on homework assignments and discuss points of confusion with the instructor and/or TAs. However, you must write your own solutions independently and you may not look at or copy any one else's solutions. You must show all of your work to receive full credit.

1. Let $z, w \in \mathbb{C}$.
   (a) Prove that $\overline{z+w} = \overline{z} + \overline{w}$.
   (b) Prove that $\overline{zw} = (\overline{z})(\overline{w})$.

2. Let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$ are not both zero.
   (a) Show that the eigenvalues of A are $\lambda = a \pm bi$.
   (b) For the following values of a and b, compute and sketch the vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A^2\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } A^3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   in $\mathbb{R}^2$.

$$(i) \quad a = 0, b = 1 \qquad (ii) \quad a = 1, b = 1 \qquad (iii) \quad a = .6, b = .6$$

   (c) It turns out that multiplication by A rotates and scales vectors. In the cases above, determine the angle and scalar by which A rotates and scales vectors. Explain your reasoning.
   (d) Now let's prove that multiplication by any such matrix A rotates and scales vectors.
      (i) Let $r = \sqrt{a^2 + b^2}$. Show that $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some θ.
      (ii) Deduce that multiplication by A rotates and scales vectors.

3. Let $S = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.
   (a) Find the eigenvalues of S. Show your work.

(b) Find unit eigenvectors corresponding to each eigenvalue of S. Show your work

(c) Use the above eigenvalues and eigenvectors to diagonalize S; that is find and invertible matrix P and a diagonal matrix D such that $S = PDP^{-1}$. Explain your reasoning.

(d) Write down the statement of the Spectral Theorem and explain why the diagonalization you found in part (c) is an orthogonal diagonalization.

4. In class we prove that a symmetric matrix S is positive definite if and only if $\mathbf{x}^T S\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$. Use this fact to show that the following matrix is not positive definite.

$$S = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 3 & 1 & 1 \\ -1 & 1 & -4 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

5. A symmetric matrix is called *negative definite* if all of its eigenvalues are negative. Show that if $\mathbf{x}^T S\mathbf{x} < 0$ for all $\mathbf{x} \neq 0$, then S is negative definite.

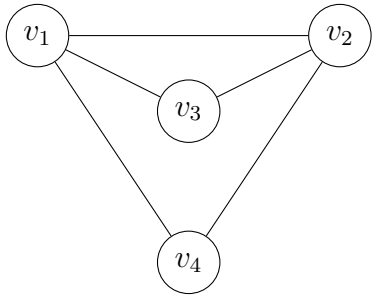
   [Note: this can be upgraded to an if and only if statement]

6. In Worksheet 9.26, we defined the incidence matrix of a graph. In this problem, we will consider a different matrix that can be defined from a graph called the **Laplacian matrix**. Let G be a graph with vertices $v_1, \ldots, v_n$. The *degree* of v_i (denoted $\deg(v_i)$) is the number of edges adjancent to v_i. The Laplacian matrix of G is the matrix whose $(i, j)-$th entry is given by

$$\begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ and } v_j \text{ share an edge} \\ 0 & \text{otherwise} \end{cases}$$

   It turns out that Laplacian matrices are positive semi-definite. We will see this in an example.

   (a) Consider the graph G below. Find the Laplacian matrix L of G.



   (b) Use Julia to show that L is positive semi-definite. Include your code and output in your solution.

7. True/False. If a statement is true, prove it. If it is false, provide a counterexample.
   (a) If S is a symmetric $n \times n$ matrix, then $\mathbb{R}^n$ has a orthonormal basis of eigenvectors of S.
   (b) Every eigenvalue of a symmetric matrix is nonnegative.
   (c) If $\lambda \neq 0$ is an eigenvalue of $A^T A$, then it is also an eigenvalue of AA^T.
   (d) If $\lambda = 0$ is an eigenvalue of $A^T A$, then it is also an eigenvalue of AA^T.